

The m -reduction in conformal field theory as the Morita equivalence on two-tori

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ABSTRACT: We study the Morita equivalence for field theories on noncommutative two-tori. For rational values of the noncommutativity parameter θ (in appropriate units) we show the equivalence between an abelian noncommutative field theory and a nonabelian theory of twisted fields on ordinary space. We concentrate on a particular conformal field theory (CFT), the one obtained by means of the m -reduction procedure [1], and show that the Morita equivalence also holds at this level. An application to the physics of a quantum Hall fluid at Jain fillings $\nu = \frac{m}{2pm+1}$ is explicitly considered in order to further elucidate such a correspondence.

KEYWORDS: Field Theories in Lower Dimensions, Nonperturbative Effects, Conformal and W Symmetry.

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1. Introduction

Noncommutative field theories (NCFT) have attracted much attention in the last years because they provide a non trivial generalization of local quantum field theories, allowing for some degree of non locality while retaining an interesting mathematical structure [2]. Indeed they naturally arise as some low energy limit of open string theory and as the compactification of M-theory on the torus [3]. Space-time noncommutativity also arises naturally when the dynamics of open strings attached to a $D2$ -brane in a B field background is considered [4]: in such a case the open strings act as dipoles of $U(1)$ gauge field of the brane and their scattering amplitudes in the low energy limit are properly described by a Super Yang-Mills gauge theory defined on a noncommutative two-torus with deformation parameter θ identified with the B field. More generally, gauge theories on tori with magnetic flux and twisted models can be reformulated in terms of noncommutative gauge theories. One motivation for the relevance of such theories is that the notion of space-time presumably has to be modified at very short distances so that the effect of the granularity of the space can be taken into account.

A significant feature of NCFT is the celebrated Morita duality between noncommutative tori [5]. This duality is a powerful mathematical result that establishes a relation, via an isomorphism, between two noncommutative algebras. Of particular importance are the algebras defined on the noncommutative torus, where it can be shown that Morita equivalence holds if the corresponding sizes of the tori and the noncommutative parameters are related in a specific way. Several results have been established in the literature about the Morita equivalence of NCFT but principally focused on noncommutative gauge theories and describing mostly classical or semiclassical aspects of them [6]. Indeed Morita duality of gauge theories on noncommutative tori is a low energy analogue of T -duality of the underlying string model [7]; when combined with the hypothesis of analyticity as a function of the noncommutativity parameter, it gives information about singular large- N

limits of ordinary $U(N)$ gauge theories [8]. Nevertheless another point of view can be usefully developed in order to establish a correspondence between NCFT and well known standard field theories. Indeed, for special values of the noncommutative parameter, one of the isomorphic theories obtained by using the Morita equivalence is a commutative field theory on an ordinary space [9].

Here we follow this line and concentrate on a particular conformal field theory (CFT), the one obtained via m -reduction technique [1], which has been recently applied to the description of a quantum Hall fluid (QHF) at Jain [10, 11] as well as paired states fillings [12, 13] and in the presence of topological defects [14]; by using Morita duality we build up the essential ingredients of the corresponding NCFT. In such a context the granularity of the space is due to the existence of a minimum area which is the result of a fractionalized magnetic flux. The m -reduction technique is based on the simple observation that, for any CFT (mother), a class of sub-theories exists, which is parameterized by an integer m with the same symmetry but different representations. The resulting theory (daughter), called Twisted Model (TM), has the same algebraic structure but a different central charge $c_m = mc$. Its application to the physics of the QHF arises by the incompressibility of the Hall fluid droplet at the plateaux, which implies its invariance under the $W_{1+\infty}$ algebra at different fillings [15], and by the peculiarity of the m -reduction procedure to provide a daughter CFT with the same $W_{1+\infty}$ invariance property of the mother theory [10, 11]. Thus the m -reduction furnishes automatically a mapping between different incompressible plateaux of the QHF. The characteristics of the daughter theory is the presence of twisted boundary conditions on the fundamental fields, which coincide with the conditions required by the Morita equivalence for a class of NCFT. Here the noncommutativity of the spatial coordinates appears as a consequence of the twisting. As a result, the m -reduction technique becomes the image in the ordinary space of the Morita duality. Furthermore the Moyal algebra, which characterizes the NCFT, has a natural realization in terms of Generalized Magnetic Translations (GMT) within the m -reduced theory when we refer to the description of a QHF at Jain fillings $\nu = \frac{m}{2pm+1}$ [10, 11]. In this paper we show how the m -reduction procedure induces a CFT which is the Morita equivalent of a NCFT by making explicit reference to the physics of a QHF at Jain fillings. We point out that in the last years there have been many investigations on the relationship between noncommutative spaces and QHF: in all such studies noncommutativity is related to the finite number N_e of electrons in a realistic sample via the rational parameter $\theta \propto \frac{1}{N_e}$, which sets the elementary area of nonlocality [16, 17]. In this context the θ -dependence of physical quantities is expected to be analytic near $\theta = 0$ because of the very smooth ultraviolet behaviour of noncommutative Chern-Simons theories [8]; as a consequence the effects of the electron's granularity embodied in θ can be expanded in powers of θ via the Seiberg-Witten map [3] and then appear as corrections to the large- N results of field theory. Our approach is quite different: the noncommutativity is present at field theory level and constrains the structure of the theory already in the large- N limit. It can be ascribed to a residual noncommutativity for large numbers of electrons which group in clusters of finite m .

The paper is organized as follows.

In section 2, we review the main steps to be performed in order to get a m -reduced CFT on the plane [1] and then we briefly recall the description of a QHF at Jain fillings $\nu = \frac{m}{2pm+1}$ as a result of the m -reduction procedure [10, 11].

In section 3, we explicitly build up the Morita equivalence between CFTs in correspondence of rational values of the noncommutativity parameter θ with an explicit reference to the m -reduced theory describing a QHF at Jain fillings. Indeed we show that there is a well defined isomorphism between the fields on a noncommutative torus and those of a non-abelian field theory on an ordinary space.

In section 4, we further clarify the deep relationship between m -reduction procedure and Morita equivalence by showing how the NCFT properties can be obtained from a generalization of the ordinary magnetic translations in a QHF context [18].

In section 5, some comments and outlooks are given.

2. The m -reduction procedure

In this section we review the basics of the m -reduction procedure on the plane (genus $g = 0$) [1] and then we show briefly how it works, referring to the description of a QHF at Jain fillings $\nu = \frac{m}{2pm+1}$ [10, 11].

In general, the m -reduction technique is based on the simple observation that for any CFT (mother) exists a class of sub-theories parameterized by an integer m with the same symmetry but different representations. The resulting theory (daughter) has the same algebraic structure but a different central charge $c_m = mc$. In order to obtain the generators of the algebra in the new theory we need to extract the modes which are divided by the integer m . These can be used to reconstruct the primary fields of the daughter CFT. This technique can be generalized and applied to any extended chiral algebra which includes the Virasoro one. Following this line one can generate a large class of CFTs with the same extended symmetry but with different central extensions. It can be applied in particular to describe the full class of Wess-Zumino-Witten (WZW) models with symmetry $\widehat{su(2)}_m$, obtaining the associated parafermions in a natural way or the incompressible $W_{1+\infty}$ minimal models [15] with central charge $c = m$. Indeed the m -reduction preserves the commutation relations between the algebra generators but modifies the central extension (i.e. the level for the WZW models). In particular this implies that the number of the primary fields gets modified.

The general characteristics of the daughter theory is the presence of twisted boundary conditions (TBC) which are induced on the component fields and are the signature of an interaction with a localized topological defect. It is illuminating to give a geometric interpretation of that in terms of the covering on a m -sheeted surface or a complex curve with branch-cuts, see for instance figure 1 for the particular case $m = 2$.

Indeed the fields which are defined on the left domain of the boundary have TBC while the fields defined on the right one have periodic boundary conditions (PBC). When we generalize the construction to a Riemann surface of genus $g = 1$, i.e. a torus, we find different sectors corresponding to different boundary conditions on the cylinder, as shown

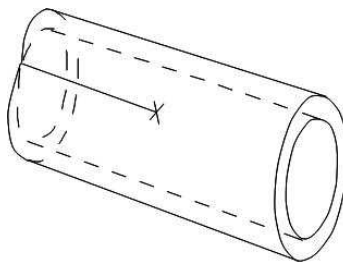


Figure 1: The edge of the 2-covered cylinder can be viewed as a separation line of two different domains of the 2-reduced CFT.

in detail in refs. [11, 13]. Finally we recognize the daughter theory as an orbifold of the usual CFT describing the QHF at a given plateau.

The physical interpretation of such a construction within the context of a QHF description is the following. The two sheets simulate a two-layer quantum Hall system and the branch cut represents TBC which emerge from the interaction with a localized topological defect on the edge [14].

Let us now briefly summarize the m -reduction procedure on the plane [1], which has been recently applied to the description of a quantum Hall fluid (QHF) at Jain [10] as well as non standard fillings [12]. Its generalization to the torus topology has been given in refs. [11, 13]. The starting point is described by a CFT with $c = 1$, in terms of a scalar chiral field compactified on a circle with general radius R^2 ($R^2 = 1$ for the Jain series [10] while $R^2 = 2$ for the non standard one [12]). Then the $u(1)$ current is given by $J(z) = i\partial_z Q(z)$, where $Q(z)$ is the compactified Fubini field with the standard mode expansion:

$$Q(z) = q - i p \ln z + \sum_{n \neq 0} \frac{a_n}{n} z^{-n}; \tag{2.1}$$

here a_n , q and p satisfy the commutation relations $[a_n, a_{n'}] = n\delta_{n,n'}$ and $[q, p] = i$. The primary fields are expressed in terms of the vertex operators $U^{\alpha_s}(z) =: e^{i\alpha_s Q(z)} :$ with $\alpha_s = \frac{s}{R}$ ($s = 1, \dots, R^2$) and conformal dimension $h = \frac{s^2}{2R^2}$.

Starting with the set of fields in the above CFT and using the m -reduction procedure we get the image of the twisted sector of a $c = m$ orbifold CFT (i.e. the TM), which describes the Lowest Landau Level (LLL) dynamics of the new filling in the QHF context. In this way the fundamental fields are mapped into m twisted fields which are related by a discrete abelian group. Indeed the fields in the mother CFT can be factorized into irreducible orbits of the discrete Z_m group, which is a symmetry of the TM, and can be organized into components, which have well defined transformation properties under this group. To compare the orbifold so built with the $c = m$ CFT, we use the mapping $z \rightarrow z^{1/m}$ and the isomorphism, defined in ref. [1], between fields on the z plane and fields on the z^m covering plane given by the following identifications: $a_{nm+l} \rightarrow \sqrt{m} a_{n+l/m}$, $q \rightarrow \frac{1}{\sqrt{m}} q$.

We perform a “double” m -reduction which consists in applying this technique into two steps.

- 1) The m -reduction is applied to the Fubini field $Q(z)$. That induces twisted boundary conditions on the currents. It is useful to define the invariant scalar field:

$$X(z) = \frac{1}{m} \sum_{j=1}^m Q(\varepsilon^j z), \tag{2.2}$$

where $\varepsilon^j = e^{i\frac{2\pi j}{m}}$, corresponding to a compactified boson on a circle with radius now equal to $R_X^2 = R^2/m$. This field describes the U(1) electrically charged component of the new filling in a QHF description.

On the other hand the non-invariant fields defined by

$$\phi^j(z) = Q(\varepsilon^j z) - X(z), \quad \sum_{j=1}^m \phi^j(z) = 0 \tag{2.3}$$

naturally satisfy twisted boundary conditions, so that the $J(z)$ current of the mother theory decomposes into a charged current given by $J(z) = i\partial_z X(z)$ and $m-1$ neutral ones $\partial_z \phi^j(z)$ [10].

- 2) The m -reduction applied to the vertex operators $U^{\alpha_s}(z)$ of the mother theory also induces twisted boundary conditions on the vertex operators of the daughter CFT. The discrete group used in this case is just the m -ality group which selects the neutral modes with a complementary cut singularity, which is necessary to reinforce the locality constraint.

The vertex operator in the mother theory can be factorized into a vertex that depends only on the $X(z)$ field:

$$\mathcal{U}^{\alpha_s}(z) = z^{\frac{\alpha_s^2(m-1)}{m}} : e^{i\alpha_s X(z)} : \tag{2.4}$$

and in vertex operators depending on the $\phi^j(z)$ fields. It is useful to introduce the neutral component:

$$\psi_1(z) = \frac{z^{\frac{1-m}{m}}}{m} \sum_{j=1}^m \varepsilon^j : e^{i\phi^j(z)} : \tag{2.5}$$

which is invariant under the twist group given in 1) and has m -ality charge $l = 1$. Then, the new primary fields are the composite vertex operators $V^{\alpha_s}(z) = \mathcal{U}^{\alpha_s}(z)\psi_l(z)$ where ψ_l are the neutral operators with m -ality charge l .

From these primary fields we can obtain the new Virasoro algebra with central charge $c = m$ which is generated by the energy-momentum tensor $T(z)$. It is the sum of two independent operators, one depending on the charged sector:

$$T_X(z) = -\frac{1}{2} : (\partial_z X(z))^2 : \tag{2.6}$$

with $c = 1$ and the other given in terms of the Z_m twisted bosons $\phi^j(z)$:

$$T_\phi(z) = -\frac{1}{2} \sum_{j,j'=1}^m : \partial_z \phi^j(z) \partial_z \phi^{j'}(z) : + \frac{m^2 - 1}{24mz^2} \tag{2.7}$$

with $c = m - 1$.

Let us notice here that, although the daughter CFT has the same central charge value, it differs in the symmetry properties and in the spectrum, depending on the mother theory we are considering (i.e. for Jain or non standard series in the case of a QHF). Now, if we choose as starting point a CFT with $c = 1$, in terms of a scalar chiral field compactified on a circle with radius $R^2 = 1$, see eq. (2.1), we get the full spectrum of excitations of a QHF at Jain fillings [10]. The dynamical symmetry is given by the $W_{1+\infty}$ algebra [19] with $c = 1$, whose generators are simply given by a power of the current $J(z)$. Indeed, by using the m -reduction procedure, we get the image of the twisted sector of a $c = m$ orbifold CFT which has $\widehat{U}(1) \times \widehat{SU}(m)_1$ as extended symmetry and describes the QHF at the new general filling $\nu = \frac{m}{2pm+1}$. The above construction has been generalized to the torus topology as well [11], confirming the physical picture just outlined.

3. Morita equivalence and the twisted model

In this section we explicitly build up the Morita equivalence between CFTs in correspondence of some rational values of the noncommutativity parameter θ with an explicit reference to the m -reduced theory describing a QHF at Jain fillings, recalled in section 2.

The Morita equivalence [5, 6] is an isomorphism between noncommutative algebras that conserves all the modules and their associated structures. Let us consider an $U(N)$ NCFT defined on the noncommutative torus T_θ^2 and, for simplicity, of radii R . The coordinates satisfy the commutation rule $[x_1, x_2] = i\theta$ [2]. In such a simple case the Morita duality is represented by the following $SL(2, Z)$ action on the parameters:

$$\theta' = \frac{a\theta + b}{c\theta + d}; \quad R' = |c\theta + d| R, \tag{3.1}$$

where a, b, c, d are integers and $ad - bc = 1$.

For rational values of the non commutativity parameter, $\theta = -\frac{b}{a}$, so that $c\theta + d = \frac{1}{a}$, the Morita transformation (3.1) sends the NCFT to an ordinary one with $\theta' = 0$ and different radius $R' = \frac{R}{a}$, involving in particular a rescaling of the rank of the gauge group [9, 8]. Indeed the dual theory is a twisted $U(N')$ theory with $N' = aN$. The classes of $\theta' = 0$ theories are parametrized by an integer m , so that for any m there is a finite number of abelian theories which are related by a subset of the transformations given in eq. (3.1).

In this context the m -reduction technique applied to the QHF at Jain fillings ($\nu = \frac{m}{2pm+1}$) can be viewed as the image of the Morita map (characterized by $a = 2p(m-1) + 1$, $b = 2p$, $c = m-1$, $d = 1$) between the two NCFTs with $\theta = 1$ and $\theta = 2p + \frac{1}{m}$ respectively and corresponds to the Morita map in the ordinary space. Indeed the $\theta = 1$ theory is an $U(1)_{\theta=1}$ NCFT while the mother CFT is an ordinary $U(1)$ theory; furthermore, when we look at the $U(1)_{\theta=2p+\frac{1}{m}}$ NCFT, its Morita dual CFT has $U(m)$ symmetry. The whole correspondence between the NCFTs and the ordinary CFTs is summarized in the following

table:

$$\begin{array}{ccc}
 U(1)_{\theta=1} & \xrightarrow[\text{(a=1,b=-1,c=0,d=1)}]{\text{Morita}} & U(1)_{\theta=0} \\
 \text{Morita} \downarrow \text{(a,b,c,d)} & & \downarrow \text{m-reduction} \\
 U(1)_{\theta=2p+\frac{1}{m}} & \xrightarrow[\text{(a=m,b=-2pm-1,c=1-m,d=2p(m-1)+1)}]{\text{Morita}} & U(m)_{\theta=0}
 \end{array} \tag{3.2}$$

It is the main result of this paper.

For more general commutativity parameters $\theta = \frac{q}{m}$ such a correspondence can be easily extended. Indeed the action of the m -reduction procedure on the number q doesn't change the central charge of the CFT under study but modifies the compactification radius of the charged sector [10, 11]. Nevertheless in this paper we are interested to the action of the Morita map on the denominator of the parameter θ which has interesting consequences on noncommutativity, so in the following we will concentrate on such an issue. The generalization of the Morita map to different rational noncommutativity parameters, and at the same time to the physics of QHF at different filling factors, will be the subject of a future publication [20].

Let us now show in detail how the twisted boundary conditions on the neutral fields of the m -reduced theory (see section 2) arise as a consequence of the noncommutative nature of the $U(1)_{\theta=2p+\frac{1}{m}}$ NCFT. In order to carry out this program let us recall that an associative algebra of smooth functions over the noncommutative two-torus T^2_θ can be realized through the Moyal product ($[x_1, x_2] = i\theta$):

$$f(x) * g(x) = \exp\left(\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})\right) f(x) * g(y) \Big|_{y=x}. \tag{3.3}$$

It is convenient to decompose the elements of the algebra, i.e. the fields, in their Fourier components. However a general field operator Φ defined on a torus can have different boundary conditions associated to any of the compact directions. For the torus we have four different possibilities:

$$\Phi(x_1 + R, x_2) = e^{2\pi i\alpha_1}\Phi(x_1, x_2), \quad \Phi(x_1, x_2 + R) = e^{2\pi i\alpha_2}\Phi(x_1, x_2), \tag{3.4}$$

where α_1 and α_2 are the boundary parameters. The Fourier expansion of the general field operator $\Phi_{\vec{\alpha}}$ with boundary conditions $\vec{\alpha} = (\alpha_1, \alpha_2)$ takes the form:

$$\Phi_{\vec{\alpha}} = \sum_{\vec{n}} \Phi^{\vec{n}} U_{\vec{n}+\vec{\alpha}} \tag{3.5}$$

where we define the generators as

$$U_{\vec{n}} \equiv \exp\left(2\pi i \frac{\vec{n} \cdot \vec{x}}{R}\right). \tag{3.6}$$

They give rise to the following Moyal commutator:

$$\left[U_{\vec{n}+\vec{\alpha}}, U_{\vec{n}'+\vec{\alpha}'}\right] = -2i \sin\left(\frac{2\pi^2\theta}{R^2}(\vec{n} + \vec{\alpha}) \wedge (\vec{n}' + \vec{\alpha}')\right) U_{\vec{n}+\vec{n}'+\vec{\alpha}+\vec{\alpha}'}, \tag{3.7}$$

where $\vec{p} \wedge \vec{q} = \varepsilon_{ij} p_i q_j$.

When the noncommutativity parameter θ takes the rational value $\theta = \frac{2qR^2}{m2\pi}$, being q and m relatively prime integers, the infinite-dimensional algebra generated by the $U_{\vec{n}+\vec{\alpha}}$ breaks up into equivalence classes of finite dimensional subspaces. Indeed the elements $U_{m\vec{n}}$ generate the center of the algebra and that makes possible for the momenta the following decomposition:

$$\vec{n}' + \vec{\alpha} = m\vec{n} + \vec{n}, \quad 0 \leq n_1, n_2 \leq m-1. \quad (3.8)$$

The whole algebra splits into equivalence classes classified by all the possible values of $m\vec{n}$, each class being a subalgebra generated by the m^2 functions $U_{\vec{n}+\vec{\alpha}}$ which satisfy the relations

$$\left[U_{\vec{n}+\vec{\alpha}}, U_{\vec{n}'+\vec{\alpha}'} \right] = -2i \sin \left(\frac{2\pi q}{m} (\vec{n} + \vec{\alpha}) \wedge (\vec{n}' + \vec{\alpha}') \right) U_{\vec{n}+\vec{n}'+\vec{\alpha}+\vec{\alpha}'}. \quad (3.9)$$

The algebra (3.9) is isomorphic to the (complexification of the) $U(m)$ algebra, whose general m -dimensional representation can be constructed by means of the following "shift" and "clock" matrices [21]:

$$Q = \begin{pmatrix} 1 & & & \\ & \varepsilon & & \\ & & \ddots & \\ & & & \varepsilon^{m-1} \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1 & 0 \\ & \cdots & \\ & & \vdots & 1 \\ 1 & & & 0 \end{pmatrix}, \quad (3.10)$$

being $\varepsilon = \exp(\frac{2\pi iq}{m})$. So the matrices $J_{\vec{n}} = \varepsilon^{n_1 n_2} Q^{n_1} P^{n_2}$, $n_1, n_2 = 0, \dots, m-1$, generate an algebra isomorphic to (3.9):

$$\left[J_{\vec{n}}, J_{\vec{n}'} \right] = -2i \sin \left(2\pi \frac{q}{m} \vec{n} \wedge \vec{n}' \right) J_{\vec{n}+\vec{n}'}. \quad (3.11)$$

Thus the following Morita mapping has been realized between the Fourier modes defined on a noncommutative torus and functions taking values on $U(m)$ but defined on a commutative space:

$$\exp \left(2\pi i \frac{(\vec{n} + \vec{\alpha}) \cdot \widehat{\vec{x}}}{R} \right) \longleftrightarrow \exp \left(2\pi i \frac{(\vec{n} + \vec{\alpha}) \cdot \vec{x}}{R} \right) J_{\vec{n}+\vec{\alpha}}. \quad (3.12)$$

As a consequence a mapping between the fields $\Phi_{\vec{\alpha}}$ is generated as follows. Let us focus, for simplicity, on the case $q = 1$ which leads for the momenta to the decomposition $\vec{n} = m\vec{n} + \vec{j}$, with $0 \leq j_1, j_2 \leq m$. The general field operator $\Phi_{\vec{\alpha}}$ on the noncommutative torus T_θ^2 with boundary conditions $\vec{\alpha}$ can be written in the form:

$$\Phi_{\vec{\alpha}} = \sum_{\vec{j}} \exp \left(2\pi i m \frac{\vec{n} \cdot \vec{x}}{R} \right) \sum_{\vec{j}=0}^{m-1} \Phi_{\vec{n}, \vec{j}} U_{\vec{j}+\vec{\alpha}}. \quad (3.13)$$

By using eq. (3.12) we obtain the Morita correspondence between fields as:

$$\Phi_{\vec{\alpha}} \longleftrightarrow \Phi = \sum_{\vec{j}=0}^{m-1} \chi^{(\vec{j})} J_{\vec{j}+\vec{\alpha}}, \quad (3.14)$$

where we have defined:

$$\chi^{(\vec{j})} = \exp\left(2\pi i \frac{(\vec{j} + \vec{\alpha}) \cdot \vec{x}}{R}\right) \sum_{\vec{q}} \Phi^{\vec{n}, \vec{j}} \exp\left(2\pi i m \frac{\vec{n} \cdot \vec{x}}{R}\right). \quad (3.15)$$

The field Φ is defined on the dual torus with radius $R' = \frac{R}{m}$ and satisfies the boundary conditions:

$$\Phi(\theta + R', x_2) = \Omega_1^+ \cdot \Phi(\theta, x_2) \cdot \Omega_1, \quad \Phi(\theta, x_2 + R') = \Omega_2^+ \cdot \Phi(\theta, x_2) \cdot \Omega_2, \quad (3.16)$$

with

$$\Omega_1 = P^b, \quad \Omega_2 = Q^{1/q}, \quad (3.17)$$

where b is an integer satisfying $am - bq = 1$. While the field components $\chi^{(\vec{j})}$ satisfy the following twisted boundary conditions:

$$\begin{aligned} \chi^{(\vec{j})}(\theta + R', x_2) &= e^{2\pi i(j_1 + \alpha_1)/m} \chi^{(\vec{j})}(\theta, x_2) \\ \chi^{(\vec{j})}(\theta, x_2 + R') &= e^{2\pi i(j_2 + \alpha_2)/m} \chi^{(\vec{j})}(\theta, x_2), \end{aligned} \quad (3.18)$$

that is

$$\left(\frac{j_1 + \alpha_1}{m}, \frac{j_2 + \alpha_2}{m}\right); \quad j_1 = 0, \dots, m-1; \quad j_2 = 0, \dots, m-1. \quad (3.19)$$

Let us observe that $\vec{j} = (0, 0)$ is the trace degree of freedom which can be identified with the $U(1)$ component of the matrix valued field or the charged component within the m -reduced theory of the QHF at Jain fillings introduced in section 2. We conclude that only the integer part of $\frac{n_i}{m}$ should really be thought of as the momentum. The commutative torus is smaller by a factor $m \times m$ than the noncommutative one; in fact upon this rescaling also the "density of degrees of freedom" is kept constant as now we are dealing with $m \times m$ matrices instead of scalars.

Summarizing, when the parameter θ is rational we recover the whole structure of the noncommutative torus and recognize the twisted boundary conditions which characterize the neutral fields (2.3) of the m -reduced theory as the consequence of the Morita mapping of the starting NCFT ($U(1)_{\theta=2p+\frac{1}{m}}$ in our case) on the ordinary commutative space. Indeed $\chi^{(0,0)}$ corresponds to the charged X field while the twisted fields $\chi^{(\vec{j})}$ with $\vec{j} \neq (0, 0)$ should be identified with the neutral ones (2.3). Therefore the m -reduction technique can be viewed as a realization of the Morita mapping between NCFTs and CFTs on the ordinary space, as sketched in the table (3.2). In the next section we further clarify such a correspondence by making an explicit reference to the QHF physics at Jain fillings. In particular we will recognize the GMT as a realization of the Moyal algebra defined in eq. (3.9).

4. Moyal algebra and generalized magnetic translations

In this section we will make explicit reference to the m -reduced theory for a QHF at Jain fillings and to the issue of GMT on a torus in order to identify in such a context a realization of the Moyal algebra.

Let us recall that all the relevant topological effects in the QHF physics, such as the degeneracy of the ground state wave function on a manifold with non trivial topology, the derivation of the Hall conductance σ_H as a topological invariant and the relation between fractional charge and statistics of anyon excitations [22, 18], can be made very transparent by using the invariance properties of the wave functions under a finite subgroup of the magnetic translation group for a N_e electrons system. Indeed their explicit expressions as the Verlinde operators [18] which generate the modular transformations in the $c = 1$ CFT are taken as a realization of topological order of the system under study [23]. In particular the magnetic translations built so far [18] act on the characters associated to the highest weight states which represent the charged statistical particles, the anyons or the electrons.

In our CFT representation of the QHF at Jain fillings [10, 11] we shall see that the Moyal algebra defined in eq. (3.9) has a natural and beautiful realization in terms of GMT. We refer to them as generalized ones because the usual magnetic translations act on the charged content of the one point functions [18]. Instead, in our TM model for the QHF (see section 2 and refs. [11, 13]) the primary fields (and then the corresponding characters within the torus topology) appear as composite field operators which factorize in a charged as well as a neutral part. Further they are also coupled by the discrete symmetry group Z_m . Then, in order to show that the characters of the theory are closed under magnetic translations, we need to generalize them in such a way that they will appear as operators with two factors, acting on the charged and on the neutral sector respectively.

Let us also recall that the incompressibility of the quantum Hall fluid naturally leads to a $W_{1+\infty}$ dynamical symmetry [19, 24]. Indeed, if one considers a droplet of a quantum Hall fluid, it is evident that the only possible area preserving deformations of this droplet are the waves at the boundary of the droplet, which describe the deformations of its shape, the so called edge excitations. These can be well described by the infinite generators W_m^{n+1} of $W_{1+\infty}$ of conformal spin $(n + 1)$, which are characterized by a mode index $m \in Z$ and satisfy the algebra:

$$\begin{aligned} \left[W_m^{n+1}, W_{m'}^{n'+1} \right] &= (n'm - nm')W_{m+m'}^{n+n'} + q(n, n', m, m')W_{m+m'}^{n+n'-2} + \dots \\ &+ d(n, m)c \delta^{n, n'} \delta_{m+m'=0} \end{aligned} \quad (4.1)$$

where the structure constants q and d are polynomials of their arguments, c is the central charge, and dots denote a finite number of similar terms involving the operators $W_{m+m'}^{n+n'-2l}$ [19, 24]. Such an algebra contains an Abelian $\widehat{U}(1)$ current for $n = 0$ and a Virasoro algebra for $n = 1$ with central charge c . It encodes the local properties which are imposed by the incompressibility constraint and realizes the allowed edge excitations [24]. Nevertheless algebraic properties do not include topological properties which are also a consequence of incompressibility. In order to take into account the topological properties

we have to resort to finite magnetic translations which encode the large scale behaviour of the QHF.

Let us consider a magnetic translation of step $(n = n_1 + in_2, \bar{n} = n_1 - in_2)$ on a sample with coordinates x_1, x_2 and define the corresponding generators $T^{n, \bar{n}}$ as:

$$T^{n, \bar{n}} = e^{-\frac{B}{4}n\bar{n}} e^{\frac{1}{2}nb^+} e^{\frac{1}{2}\bar{n}b}, \quad (4.2)$$

where $b^+ = i\partial_{\bar{\omega}} - i\frac{B}{2}\omega$, $b = i\partial_{\omega} + i\frac{B}{2}\bar{\omega}$, ω is a complex coordinate ($\bar{\omega}$ being its conjugate) and B is the transverse magnetic field. They satisfy the relevant property:

$$T^{n, \bar{n}} T^{m, \bar{m}} = q^{-\frac{n \times m}{4}} T^{n+m, \bar{n}+\bar{m}}, \quad (4.3)$$

where q is a root of unity.

Furthermore, it can be easily shown that they admit the following expansion in terms of the generators W_{k-1}^{l-1} of the $W_{1+\infty}$ algebra:

$$T^{n, \bar{n}} = e^{-\frac{B}{4}n\bar{n}} \sum_{k,l=0}^{\infty} (-)^l \frac{n^k}{2^k n} \frac{\bar{n}^l}{2^l \bar{n}} W_{k-1}^{l-1}, \quad (4.4)$$

where now the local $W_{1+\infty}$ symmetry and the global topological properties are much more evident because the coefficients in the above series depend on the topology of the sample.

Within our m -reduced theory for a QHF at Jain fillings [10][11] it can be shown that also magnetic translations of step (n, \bar{n}) decompose in equivalence classes and can be factorized into a group, with generators $T_C^{n, \bar{n}}$, which acts only on the charged sector as well as a group, with generators $T_S^{j_i, \bar{j}}$, acting only on the neutral sector [25]. The presence of the transverse magnetic field B reduces the torus to a noncommutative one and the flux quantization induces rational values of the noncommutativity parameter θ . As a consequence the neutral magnetic translations realize a projective representation of the $su(m)$ algebra generated by the elementary translations:

$$J_{a,b} = e^{-2\pi i \frac{ab}{m}} T_S^{a,0} T_S^{0,b}; \quad a, b = 1, \dots, m, \quad (4.5)$$

which satisfy the commutation relations:

$$[J_{a,b}, J_{\alpha,\beta}] = -2i \sin\left(\frac{2\pi}{m}(a\beta - b\alpha)\right) J_{a+\alpha, b+\beta}. \quad (4.6)$$

The GMT operators above defined (see eqs. (4.2) and (4.5)) are a realization of the Moyal operators introduced in eq. (3.6) and the algebra defined by eq. (4.6) is isomorphic to the Moyal algebra given in eqs. (3.9) and (3.11). Such operators generate the residual symmetry of the m -reduced CFT which is Morita equivalent to the NCFT with rational non commutativity parameter $\theta = 2p + \frac{1}{m}$.

5. Conclusions and outlooks

In conclusion, the Morita equivalence gives rise to an isomorphism between noncommutative algebras but, in the special case of a rational noncommutativity parameter θ , one of

the isomorphic theories is a commutative one [9]. In this paper we have shown by means of the Morita equivalence that a NCFT with $\theta = 2p + \frac{1}{m}$ is mapped to a CFT on an ordinary space. We identified such a CFT with the m -reduced CFT developed in [10, 11] for a QHF at Jain fillings, whose neutral fields satisfy twisted boundary conditions. In this way we gave a meaning to the concept of "noncommutative conformal field theory", as the Morita equivalent version of a CFT defined on an ordinary space. The image of Morita duality in the ordinary space is given by the m -reduction technique and the Moyal algebra which reflects the noncommutative nature is realized by means of GMT. Furthermore a new relation between noncommutative spaces and QHF physics has been established within a perspective which is very different from the ones developed in the literature in the last years [16, 17]. The generalization of the Morita map to different rational noncommutativity parameters θ , and at the same time to the physics of QHF at paired state filling factors, will be the subject of a future publication [20]. That will help us to shed new light also on the connections between a system of interacting D -branes and the physics of quantum Hall fluids [26]. Recently the twisted CFT approach provided by m -reduction has been successfully applied to Josephson junction ladders and arrays of non trivial geometry in order to investigate the existence of topological order and magnetic flux fractionalization in view of the implementation of a possible solid state qubit protected from decoherence [27] as well as to the study of the phase diagram of the fully frustrated XY model on a square lattice [28]. So, it could be interesting to further elucidate the topological properties of Josephson systems with non trivial geometry and the consequences imposed by the request of space-time noncommutativity by means of Morita mapping. Finally, the generality of the m -reduction technique allows us to extend many of the present results to $D > 2$ theories as string and M -theory.

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